Instantons

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Abstract

In this lectures, I will introduce the instanton solution of QCD [1–4]

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I. WINDING NUMBER

Let us QCD action.

$$\begin{aligned} S_{Min} &= \int dtd^3x\Big(-\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\Big) \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \\ D_\mu &= \partial_\mu - igA_\mu \end{aligned} \tag{1}$$

The Euclidean QCD is obtained by the following rules

$$\begin{aligned} k_0 &= ik_4, \quad t = -ix_4 \\ S_{Eucl} &= -iS_{Min} \\ A_\mu &= \frac{g\tau^a}{2i}A^a_\mu \end{aligned} \tag{2}$$

from which the Euclidean action becomes,

$$\begin{aligned} S_{Eucl} &= \int d^4x\Big(-\frac{1}{2g^2}\mathrm{Tr}[G_{\mu\nu}G_{\mu\nu}]\Big) \\ G_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \end{aligned} \tag{3}$$

Now we want a finite action solution

$$G_{\mu\nu} \overset{r=\infty}{\to} 0 \tag{4}$$

This means that the gauge field should become a pure gauge at the boundary.

$$A_\mu(r\to\infty) = U\partial_\mu U^{-1}(\alpha_1, \alpha_2, \alpha_3) \tag{5}$$

Since U is an SU(2) value function, A_μ at eq.(5) is a mapping

$$S_3^{phys} \to S_3^{SU(2)-int} \tag{6}$$

which is characterized by the winding number $\Pi_n(S_n) = Z$. This number is called the Pontryagin index

$$Q = -\frac{1}{16\pi^2}\int d^4x \mathrm{Tr}[\widetilde{G}G] \tag{7}$$

where, $\widetilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G_{\alpha\beta}$. The factor of 1/2 is to ensure that $\widetilde{\widetilde{G}}_{\mu\nu} = G_{\mu\nu}$

One can prove that

$$D_\mu \widetilde{G}_{\mu\nu} = 0 \tag{8}$$

This Bianchi type identities can be proven by using the following identity

$$\epsilon_{\mu\nu\alpha\beta}D_\mu D_\alpha D_\beta = -\epsilon_{\mu\nu\alpha\beta}D_\alpha D_\mu D_\beta = \epsilon_{\mu\nu\alpha\beta}D_\alpha D_\beta D_\mu \tag{9}$$

Using the Theorem in Eq. (8), one can show that

$$Q = \int d^4x \partial_\mu j_\mu = \oint_{S_3^{phy}} d\sigma_\mu j^\mu \tag{10}$$

where

$$j_\mu = -\frac{1}{8\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}[A_\nu(\partial_\alpha A_\beta + \frac{2}{3}A_\alpha A_\beta)] \tag{11}$$

A. Prototype

For finite action solution, one can use the following identity at the boundary $r \to \infty$.

$$G_{\alpha\beta} = 0 \to \partial_\alpha A_\beta = -A_\alpha A_\beta \tag{12}$$

Then

$$\begin{aligned} A &= -\frac{1}{8\pi^2}\epsilon_{\mu\nu\alpha\beta}\int d\sigma_\mu \mathrm{Tr}[A_\nu(\partial_\alpha A_\beta + \frac{2}{3}A_\alpha A_\beta)] \\ &= -\frac{1}{24\pi^2}\epsilon_{\mu\nu\alpha\beta}\int d\sigma_\mu \mathrm{Tr}[(\partial_\nu U)U^{-1}(\partial_\alpha U)U^{-1}(\partial_\beta U)U^{-1}] \end{aligned} \tag{13}$$

To check that the normalization is correct, consider a prototype

$$\begin{aligned} U_1(x) &= (x_4 + ix_i\sigma_i)/|x| = \hat{x}_\mu s_\mu \\ U_1^{-1}(x) &= (x_4 - ix_i\sigma_i)/|x| = \hat{x}_\mu \bar{s}_\mu \\ UU^{-1} &= \hat{x}\cdot s\hat{x}\cdot\bar{s} = 1 \end{aligned} \tag{14}$$

where $s_\mu = (1, i\vec{\sigma})$.

Some useful identities are

$$\begin{aligned} \partial_\mu U &= \frac{s_\mu - \hat{x}_\mu\hat{x}\cdot s}{|x|} \\ (\partial_\mu U)U^{-1} &= \frac{s_\mu\hat{x}\cdot\bar{s} - \hat{x}_\mu}{|x|} = 2i\Sigma_{\mu\nu}\frac{\hat{x}_\nu}{|x|} \\ U^{-1}(\partial_\mu U) &= \frac{\hat{x}\cdot\bar{s}s_\mu - \hat{x}_\mu}{|x|} = -2i\bar{\Sigma}_{\mu\nu}\frac{\hat{x}_\nu}{|x|} \end{aligned} \tag{15}$$

where

$$\begin{aligned} \Sigma_{\mu\nu} &= \frac{-i}{2}\Big(s_\mu\bar{s}_\nu - \delta_{\mu\nu}\Big) = \frac{-i}{4}\Big(s_\mu\bar{s}_\nu - s_\nu\bar{s}_\mu\Big) = \eta^{i\mu\nu}\sigma_i/2 \\ \bar{\Sigma}_{\mu\nu} &= \frac{i}{2}\Big(\bar{s}_\nu s_\mu - \delta_{\mu\nu}\Big) = \frac{i}{4}\Big(\bar{s}_\nu s_\mu - \bar{s}_\mu s_\nu\Big) = \bar{\eta}^{i\mu\nu}\sigma_i/2 \end{aligned} \tag{16}$$

where

$$\eta^{i\mu\nu} = -\eta^{i\nu\mu} = \{ \begin{matrix} \epsilon^{i\mu\nu} & \text{for} & \mu,\nu = 1,2,3 \\ +\delta^{i\mu} & \text{for} & \nu = 4 \end{matrix} \tag{17}$$

$$\bar{\eta}^{i\mu\nu} = -\bar{\eta}^{i\nu\mu} = \{ \begin{matrix} \epsilon^{i\mu\nu} & \text{for} & \mu,\nu = 1,2,3 \\ -\delta^{i\mu} & \text{for} & \nu = 4 \end{matrix} \tag{18}$$

The properties of the ϵ tensors, useful for manipulating the instantons, are given in the appendix of [2]

Some useful identities are,

$$s_\mu\hat{x}\cdot\bar{s} = -\hat{x}\cdot s\bar{s}_\mu + 2\hat{x}_\mu \tag{19}$$

substituting this, we find

$$Q = 1 \tag{20}$$

II. THE YANG-MILLS INSTANTONS

Using,

$$
\begin{aligned}
-\int d^4x \mathrm{Tr}[(G \pm \widetilde{G})^2] \geq 0 \\
\to -\int \mathrm{Tr}[G^2] \geq \mp \int d^4x \mathrm{Tr}[\widetilde{G}G] \\
S \geq (8\pi^2/g^2)|Q|
\end{aligned}
\tag{21}
$$

equality holding when $\widetilde{G} = G$. That is to say, such gluon field configurations satisfies the equations of motions, $D_\mu G^{\mu\nu} = 0$.

One solution is

$$
A_\mu = -2i\Sigma_{\mu\nu} \frac{(x-a_1)_\nu}{|(x-a_1)|^2 + \lambda_1^2} \tag{22}
$$

This solution at $r \to \infty$ goes to, $A_\mu = -2i\Sigma_{\mu\nu}\frac{\hat{x}_\nu}{|x|}$.

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