Symmetry breaking in vector like gauge theories

Su Houng Lee[1,*]

*[1]IPAP and Department of Physics,
Yonsei University, Seoul 120-749 Korea*

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Abstract

In this lectures, I will discuss restrictions on symmetry breaking in vector like gauge theories [1]

PACS numbers:

*Electronic mail:`suhoung@phya.yonsei.ac.kr`

I. NOTATION: EUCLIDEAN FORMULATION

The QCD partition function is given by

$$Z = \int [dA_\mu(x)][dq(x)d\bar{q}(x)] \exp[iS_{QCD}], \tag{1}$$

where

$$S_{QCD} = \int d^4x \Big[-\frac{1}{2g^2}F^2 + \bar{q}(i\gamma^\mu(\partial_\mu - iA_\mu) - m_q)q \Big] \tag{2}$$

The Wick rotation to imaginary time,

$$x_0 = t \rightarrow -i\tau = -ix_4 \tag{3}$$

comes with the following,

$$\begin{aligned} d^4x &\rightarrow -id^4x_e && (4)\\ A_0 &\rightarrow iA_4 && (5)\\ F_{\mu\nu}F^{\mu\nu} = -F_{0i}^2 + F_{ij}^2 &\rightarrow F_{4i} + F_{ij}^2 && (6)\\ i\gamma^\mu(\partial_\mu - iA_\mu) &\rightarrow i\gamma_0(\partial_0 - iA_0) - i\gamma_i(\partial_i - iA_i) \\ &\rightarrow -\gamma_0(\partial_4 - iA_4) - i\gamma_i(\partial_i - iA_i) = -\gamma_\mu^E(\partial_\mu - igA_\mu), && (7) \end{aligned}$$

where $\gamma_4^E = \gamma_0, \gamma_i^E = i\gamma_i$ to satisfy

$$\{\gamma_\mu^E, \gamma_\nu^E\} = \delta_{\mu\nu} \tag{8}$$

Then the partition function becomes

$$Z = \int [dA_\mu(x)][dq(x)d\bar{q}(x)] \exp[-S_{QCD}^E], \tag{9}$$

where

$$S_{QCD}^E = \int d^4x_E \Big[\frac{1}{2g^2}F^2 + \bar{q}(\gamma_\mu^E(\partial_\mu - iA_\mu) + m_q)q \Big]. \tag{10}$$

We will call $\partial_\mu - iA_\mu = D_\mu$.

A. Banks-Casher formula

We will first prove the Banks-Casher formula [2] extensively used in numerical simulations to calculate chiral symmetry breaking.

After integrating over the quark field, the partition function in Eq. 9 can be written as,

$$\begin{aligned} Z &= \int [dA_\mu(x)] \det[\not{D} + m] \exp[-\frac{1}{2g^2}F^2] \\ &= \int [d\mu] \end{aligned} \tag{11}$$

First, one should note that $d\mu$ is a positive definite measure. To prove this, one notes that the eigenvalues of the determinant comes in pairs. If ψ is a eigenvector $i\not{D}\psi = \lambda\psi$, then so is $\gamma^5\psi$ because $i\not{D}\gamma^5\psi = -i\gamma^5\not{D}\psi = -\lambda\gamma^5\psi$. Hence,

$$\det[\not{D} + m] = \prod_{\lambda}(m - i\lambda) = \prod_{\lambda>0}(m^2 + \lambda^2) > 0 \tag{12}$$

Now one can work out the chiral order parameter, which goes as,

$$\langle\bar{q}q\rangle \ = \int[d\mu]\langle\bar{q}q\rangle^A \tag{13}$$

Now, it can be shown that

$$\begin{aligned}\langle\bar{q}q\rangle^A = \frac{1}{V}\int d^4x\langle\bar{q}q(x)\rangle^A &= \frac{1}{V}\int d^4x\langle x|\frac{1}{\not{D}+m}|x\rangle \\ &= \sum_{\lambda} d\lambda\frac{|\psi_\lambda(x)|^2}{V}\frac{1}{m-i\lambda} \\ &= \int d\lambda\rho(\lambda)\frac{1}{m-i\lambda}.\end{aligned} \tag{14}$$

To discuss chiral symmetry breaking in the chiral limit, we have to look at $\lim_{m\to 0}\langle\bar{q}q\rangle$. To this end, we note, $\lim_{m\to 0}\frac{1}{m-i\lambda} = i\mathrm{P}(\frac{1}{\lambda}) + \pi\delta(\lambda)$ so that,

$$\langle\bar{q}q\rangle^A = \pi\rho(0) \tag{15}$$

1. Isospin symmetry

$$\begin{aligned}\langle\bar{u}u - \bar{d}d\rangle^A = &= \int d\lambda\rho(\lambda)\Big(\frac{1}{m_u - i\lambda} - \frac{1}{m_d - i\lambda}\Big) \\ &= (m_d - m_u)\int_{-\infty}^{\infty} d\lambda\rho(\lambda)\frac{1}{(m_u - i\lambda)(m_d - i\lambda)}.\end{aligned} \tag{16}$$

The integral is finite at $m_u = m_d$ limit since the singularity is outside the real axis. Since the limit $m_u = m_d \to 0$ is continuous the same vacuum should be taken and hence vector symmetry is not broken.

Another point which can be taken to prove the theorem is using the lattice regularization method. Suppose there are n fermion component per lattice sites. Then in a volume V there are $(V/a^4)n$ Fermi components. The Dirac operator is a $[(V/a^4)n] \times [(V/a^4)n]$ matrix, there should be $(V/a^4)n$ eigenvalues. Hence,

$$\begin{aligned}|\langle\bar{u}u - \bar{d}d\rangle^A| = &= |(m_d - m_u)|\int_{-\infty}^{\infty} d\lambda\rho(\lambda)\frac{1}{(m_u - i\lambda)(m_d - i\lambda)} \\ &= \frac{|(m_d - m_u)|}{m_u m_d}\int_{-\infty}^{\infty} d\lambda\rho(\lambda) \\ &= \frac{|(m_d - m_u)|}{m_u m_d}\frac{n}{a^4}.\end{aligned} \tag{17}$$

Vector symmetry that can be put on the lattice are unbroken.

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[1] C. Vafa and E. Witten, Nucl. Phys. **B 234** 173 (1984).

[2] T. Banks and A. Casher, Nucl. Phys. **B 169** 103 (1980).