Skyrmion

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Abstract

In this lectures, I will introduce Skyrmions[1].

PACS numbers:

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I. THE LAGRANGIAN

Let us again start with the Chiral Lagrangian for U.

$$L = \frac{1}{4} f_\pi^2 \mathrm{Tr}\Big(\partial_\mu U \partial^\mu U^\dagger\Big) + L_2, \tag{1}$$

which transforms under

$$U(2)_L \bigotimes U(2)_R. \tag{2}$$

as follows

$$U' = g_L U g_R^\dagger \tag{3}$$

The stabilizing term L_2 is needed for stabilizing the Soliton solution, and is taken to be

$$L_2 = \frac{1}{32e^2} \mathrm{Tr}\Big([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2\Big), \tag{4}$$

where e is a dimensionless constant.

A. Topological Solitons

Skyrme noted that for a static field configuration, U is a map from R^3 of space into the group manifold of SU(2).

$$U(x) = \exp[2i\pi(x)/f_\pi] \tag{5}$$

In particular for a finite energy solution U has to become the same constant in all direction at $r \to \infty$. In such cases, it becomes a mapping from $S^3 \to$ SU(2) manifold. These mapping are topologically nontrivial and are characterized by winding numbers,

$$\Pi_3(SU(2)) = Z. \tag{6}$$

This number cannot evolve continuously in time from one value to another, so a field configuration of non-zero winding number is topologically stable. This winding number can be quantified in a topological current,

$$B^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}[L_\nu L_\alpha L_\beta], \tag{7}$$

where

$$L_\mu = U^\dagger \partial_\mu U. \tag{8}$$

Scaling arguments can be used to demonstrate the stability of a topological soliton. For that purpose, suppose we have a topological solution that gives the following mass,

$$M = -\int d^3x \Big(-\frac{f_\pi^2}{4} \mathrm{Tr}[L_i L_i] - \frac{1}{32e^2} \mathrm{Tr}[[Li, L_j]^2]\Big) \tag{9}$$

. Scale the solution as follows,

$$U(x) \rightarrow V(x/\lambda) \tag{10}$$

We find that the two terms of the lagrangian scale as,

$$M = \lambda M_1 + \frac{1}{\lambda} M_2 \tag{11}$$

Hence are stable.

B. Skyrmion

The stable soliton solution satisfies the field equation and has the boundary condition,

$$U(r = \infty) = 1 \tag{12}$$

The famous Hedgehog ansatz is,

$$U_0(x) = \exp[i\tau_a \hat{x}_a F(x)] \tag{13}$$

with boundary condition $F(0) = \pi$ and $F(\infty) = 0$. This satisfies the boundary condition on U and gives winding number 1.

C. Baryon number current

The identification of topological winding number to Baryon number was noted by Witten[3].

1. Parity transformation

The Lagrangian in Eq.(1) has the following naive separate symmetries Parity transfor-

TABLE I: List.

Transformation	alternative	definition
$x \rightarrow -x$		P_0
$\pi(x) \rightarrow -\pi(x)$	$U \rightarrow U^{-1}$	$(-1)^{n_\pi}$
$U \rightarrow U^T$		charge -conjugation

mation is defined as $P_0(-1)^{n_\pi}$. Hence, the lagrangian in Eq. (1) has redundant symmetry $P - 0$ and $(-1)^{n_\pi}$.

Witten[3] has realized that one can add anomalous interaction term that preserves parity, as in QCD, but breaks the redundant symmetry appearing in Eq.(1). This additional term is found to be quantized, as angular momentum or the quantization condition of charge around a magnetic monopole.

2. An example

Suppose that there is a particle moving on top of a unit sphere. The lagrangian is $L = \frac{1}{2}m\int dt\dot{x}_i^2$, which gives the equation of motion $m\ddot{x}_i + mx_i(\sum_k \dot{x}_k^2) = 0$ with constraint $\sum x_i^2 = 1$. This has redundant symmetry in $t \leftrightarrow -t$ and $x \leftrightarrow -x$ separately. But we can have something like a Coriolis force or a charge particle motion in the presence of a magnetic monopole at the center. Then the new equation of motion is,

$$m\ddot{x}_i + mx_i(\sum_k \dot{x}_k^2) = \alpha\epsilon_{ijk}x_j\dot{x}_k \tag{14}$$

The Action to this problem is the following, assuming $\nabla \times \mathbf{A} = \mathbf{x}/\mathbf{x^3}$, the action is,

$$I = \int (\frac{1}{2}\dot{x}_i^2 + \alpha A_i\dot{x}_i)dt \tag{15}$$

but we can use Gauss's law to write

$$\exp\left(i\alpha\int_\gamma A_i dx^i\right) = \exp\left(i\alpha\int_D F_{ij}d\Sigma^{ij}\right) \tag{16}$$

which gives the quantization condition,

$$\exp\left(i\alpha\int_\gamma A_i dx^i\right) = \exp\left(i\alpha\int_D F_{ij}d\Sigma^{ij}\right) \tag{17}$$

3. QCD

As with the simple example above, we can add a term that breaks the redundant symmetry. It is

$$\Gamma = \int_Q \omega_{ijklm}d\Sigma^{ijklm} \tag{18}$$

As before, this leads to the following quantization condition

$$\int_{S_5} \omega_{ijklm}d\Sigma^{ijklm} = 2\pi \cdot \text{integer} \tag{19}$$

takin the "integer " to be one, we can normalize the ω_{ijklm}. Therefore the final action is

$$I = \int [L_1 + L_3] + n\Gamma \tag{20}$$

One can show that

$$n\Gamma = n\frac{2}{15\pi^2 F_\pi^5}\int d^4x\epsilon\mu\nu\alpha\beta\mathrm{Tr}[A\partial_\mu A\partial_\nu A\partial_\alpha A\partial_\beta A] + \textit{higher order terms} \tag{21}$$

where $A = \pi^a\tau^a$.

Now, by gauging the external charge, we can identify the charges. Then gauging it to the electro magnetic charge, and comparing to the Anomalous $\pi^0 \to \gamma\gamma$ decay, we find $n = N_c$. We can also identify B_μ as the Baryon number current.

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