Spontaneous Symmetry breaking

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(Dated: September 14, 2010)

Abstract

In this lectures, I will talk about a concrete example of spontaneous symmetry breaking. Effective potential and gap equation in NJL model.

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I. SPONTANEOUS SYMMETRY BREAKING: A FAMOUS EXAMPLE

After Goldtone's paper in 1961[1], much work have been done to understand the mechanism of spontaneous symmetry breaking. Among a number of renounced papers[2–4], I will summarize the beginning part of ref.[3].

A. The effective potential

The connected generating functional is

$$\begin{aligned} e^{iW(J)} &= \langle 0^+|0^-\rangle_J, \\ W &= \sum_n \frac{1}{n!}\int d^4x_1\cdot\cdot d^4x_n G^{(n)}(x_1,...,x_n)J(x_1)...J(x_n) \end{aligned} \tag{1}$$

The classical field is defined as

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} \tag{2}$$

Then the effective action, defined by functional Legendre transformation

$$\begin{aligned} \Gamma(\phi_c) &= W(J) - \int d^4x J(x)\phi_c(x) \\ &= \sum_n \frac{1}{n!}\int d^4x_1\cdot\cdot d^4x_n \Gamma^{(n)}(x_1,...,x_n)\phi_c(x_1)...\phi_c(x_n) \end{aligned} \tag{3}$$

generates 1PI (one particle irreducible) Green's function. For constant field, it becomes equal to -potential,

$$\Gamma = \int d^4x[-V(\phi_c) + \frac{1}{2}(\partial_\mu\phi_c)^2..] \tag{4}$$

By analyzing the effective potential, we can learn about the vacuum.

B. Loop expansion

Let us introduce a parameter a into the Lagrangian density, by defining,

$$\mathcal{L}(\phi, \partial_\mu\phi, a) \equiv a^{-1}\mathcal{L}(\phi, \partial_\mu\phi) \tag{5}$$

Loop expansion is equivalent to a power-series expansion in a. Let P be the power of a graph. Then, it is easy to see that

$$P = I - V \tag{6}$$

where I is the number of internal line and V the number of vertex. Note for a propagator comes from the inverse of the quadratic term of the Lagrangian and is thus proportional to a. The vertex comes from the interaction part of the Lagrangian, and is proportional to a^{-1}.

One the other hand, one can also show that the number of loops, L is given by

$$L = I - V + 1, \tag{7}$$

because the number of loop is equal to independent momentum integral. The internal line carries one momentum, while the vertex takes away one by a delta function, except one delta function for overall momentum conservation.

Hence,

$$P = L - 1 \tag{8}$$

Loop expansion does not change under shift of fields and is ideal for investigating the vacuum structure.

C. A Sample Lagrangian

Consider a simple toy model.

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4 \\ & \frac{1}{2}A(\partial_\mu \phi)^2 + \frac{1}{2}B\phi^2 + \frac{1}{4!}C\phi^4 \end{aligned} \tag{9}$$

where, A, B, C are wave-function, mass and coupling-constant renormalization counterterms to be determined self-consistenly, order by order in the expansion, by imposing the definitions of the scale of the renormalized field, the renormalized mass, and the renormalized coupling constants.

To lowest order,

$$V = \frac{\lambda}{4!}\phi_c^4 \tag{10}$$

To one loop order,

$$\begin{aligned} V &= \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}B\phi_c^2 + \frac{1}{4!}C\phi_c^4 \\ &\quad + i\int \frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\frac{1}{2}\lambda\phi_c^2}{k^2 + i\epsilon}\right)^n . \end{aligned} \tag{11}$$

Here, the factors come from

1. $\frac{1}{2}$ in front of λ comes from the symmetry of two ϕ_c in a group.
2. The 2 in $\frac{1}{2n}$ comes from the symmetry of the loop and n from the symmetry of the beginning.

After Wick rotation

$$\int_{-\infty}^{\infty} dk_0 = i\int_{\infty}^{\infty} dk_4 \tag{12}$$

the second line of eq.(11), would be

$$-\int \frac{d^4k_E}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\Big(\frac{\frac{1}{2}\lambda\phi_c^2}{-k_E^2}\Big)^n \tag{13}$$

Each term in the infinite sum is highly divergent. However the infinite sum can be performed and gives,

$$\begin{aligned} V &= \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}B\phi_c^2 + \frac{1}{4!}C\phi_c^4 \\ &+\frac{1}{2}\int \frac{d^4k_E}{(2\pi)^4}\ln\Big(1+\frac{\lambda\phi_c^2}{2k_E^2}\Big). \end{aligned} \tag{14}$$

After regularizing the integral with a ultraviolet cut off,

$$\begin{aligned} V &= \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}B\phi_c^2 + \frac{1}{4!}C\phi_c^4 \\ &+\frac{\lambda\Lambda^2}{64\pi^2}\phi_c^2 + \frac{\lambda^2\phi_c^4}{256\pi^2}\Big(\ln\frac{\lambda\phi_c^2}{2\Lambda^2}-\frac{1}{2}\Big). \end{aligned} \tag{15}$$

From the renormalized mass condition,

$$\frac{d^2V}{d\phi_c^2}|_0 = 0, \longrightarrow B = -\frac{\lambda\Lambda^2}{32\pi^2} \tag{16}$$

To avoid singularity, we introduce a mass scale M and renormalize the coupling at,

$$\frac{d^4V}{d\phi_c^4}|_M = \lambda, \longrightarrow C = -\frac{3\lambda^2}{32\pi^2}\Big(\ln\frac{\lambda M^2}{2\Lambda^2}+\frac{11}{3}\Big) \tag{17}$$

Adding all these, give

$$V = \frac{\lambda}{4!}\phi_c^4 + \frac{\lambda^2\phi_c^4}{256\pi^2}\Big(\ln\frac{\phi_c^2}{M^2}-\frac{25}{6}\Big) \tag{18}$$

II. SPONTANEOUS SYMMETRY BREAKING: GAP EQUATION IN NJL MODEL

A. Basics

Consider the following transformation.

$$q' = e^{i(\alpha+\beta\gamma_5)} q \tag{19}$$

Here, α, β could be U(N_f) valued. It should be noted that while the transformation involving α is a group transformation, that involving β is not. It can be shown, however, that if one applies this transformation to either the left or right handed transformation, they become two independent transformation.

$$q'_L = \frac{1}{2}(1-\gamma_5)q' = e^{i(\alpha-\beta)} q_L$$
$$q'_R = \frac{1}{2}(1+\gamma_5)q' = e^{i(\alpha+\beta)} q_L. \tag{20}$$

That is to say,

$$(1+\gamma_5)e^{i(\alpha+\beta\gamma_5)} = e^{i(\alpha+\beta)}(1+\gamma_5)$$
$$(1-\gamma_5)e^{i(\alpha+\beta\gamma_5)} = e^{i(\alpha-\beta)}(1-\gamma_5) \tag{21}$$

or

$$e^{i(\alpha+\beta\gamma_5)} = \frac{1}{2}(e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)}) + \frac{1}{2}\gamma_5(e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)})$$
$$e^{-i(\alpha-\beta\gamma_5)} = \frac{1}{2}(e^{-i(\alpha-\beta)} + e^{-i(\alpha+\beta)}) + \frac{1}{2}\gamma_5(e^{-i(\alpha-\beta)} - e^{-i(\alpha+\beta)}) \tag{22}$$

Therefore,

$$\theta_L = \alpha - \beta$$
$$\theta_R = \alpha + \beta \tag{23}$$

are group transformations and are the $\mathrm{U}_L(N_F) \bigotimes U_R(N_R)$ transformations.

B. NJL model

Let us consider the following simplified NJL model with $N_F = 1$[8].

$$L = \bar{q}i\partial\!\!\!/ q + g[(\bar{q}q)^2 + (\bar{q}i\gamma_5 q)^2], \tag{24}$$

where we assume a summation over the number of colors N_c.

Using the transformation properties of Eq.(19) or Eq.(20) and Eq.(22), we find

$$(\bar{q}q) = (\bar{q}q)\cos(2\beta) + (\bar{q}i\gamma_5 q)\sin(2\beta)$$
$$(\bar{q}i\gamma_5 q) = -(\bar{q}q)\sin(2\beta) + (\bar{q}i\gamma_5 q)\cos(2\beta) \tag{25}$$

Therefore, the lagrangian in Eq.(24) is invariant under chiral transformation.

1. *The mean field*

In the mean field approximation, the vacuum expectation value of the lagrangian should be of the following,

$$\begin{aligned} \langle(\bar{q}q)^2\rangle &= \langle\bar{q}q\rangle^2 - \frac{1}{N_c}\langle\bar{q}q\rangle\langle\bar{q}q\rangle \\ \langle(\bar{q}i\gamma_5 q)^2\rangle &= 0 + \frac{1}{N_c}\langle\bar{q}q\rangle\langle\bar{q}q\rangle \end{aligned} \tag{26}$$

Therefore the Lagrangian in the mean field approximation is

$$\begin{aligned} L &= \bar{q}i\partial\!\!\!/ q + 2g\langle\bar{q}q\rangle\bar{q}q - g\langle\bar{q}q\rangle^2 \\ &= \bar{q}i\partial\!\!\!/ q - M\bar{q}q - \frac{1}{2}m_0^2\sigma_0^2 \end{aligned} \tag{27}$$

where we defined,

$$\begin{aligned} M &= -2g\langle\bar{q}q\rangle = G\sigma_0, \quad or \quad \sigma_0 = -(G/m_0^2)\langle\bar{q}q\rangle \\ 2g &= (G/m_0)^2 \end{aligned} \tag{28}$$

2. *Gap equation*

The gap equation is obtained from the mean field approximation,

$$\begin{aligned} -(2g)^{-1}M = \langle\bar{q}q\rangle &= -iN_c \lim_{x\to 0+} \mathrm{Tr}S_F(x;M) \\ &= 2N_C \int_{p<\Lambda} \frac{d^3p}{(2\pi)^3}\frac{-M}{\sqrt{M^2+p^2}} \\ &= -\frac{N_c\Lambda^3}{2\pi^2} x\Big(\sqrt{1+x^2} - x^2\ln\frac{1+\sqrt{1+x^2}}{x}\Big)_{x=M/\Lambda} \end{aligned} \tag{29}$$

We could also have done the integral in 4 dimensions using

$$\int_{-\infty}^{\infty} dp_0 = i\int_{-\infty}^{\infty} dp_4 \tag{30}$$

Then we have,

$$\begin{aligned} \langle\bar{q}q\rangle &= -N_c\int_{p_E^2<\Lambda_4} \frac{d^4p_E}{(2\pi)^4}\frac{4M}{p_E^2+M^2} \\ &= -\frac{N_c\Lambda_4^3}{4\pi^2} x\Big(1 - x^2\ln(1+1/x^2)\Big)_{x=M/\Lambda_4} \end{aligned} \tag{31}$$

The solution of Eq.(29) has two solutions, one is the trivial solution $M=0$ and the other has a solution when,

$$1 - \frac{2gN_c}{\pi^2}\int_0^{\Lambda}\frac{p^2dp}{E_p} = 0 \tag{32}$$

This has a non trivial solution with finite M when g is larger than

$$g_c \equiv \pi^2/N_C\Lambda^2 \tag{33}$$

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